Chapter 2
First and Second Order Systems

In this chapter, the properties of the transfer function and frequency response of first and second order systems are studied on some examples from electrical circuit laws. We show that their properties are governed by the poles (i.e., the zeros of the denominator) of the transfer function which is a rational fraction. A geometric argument based on the location of the poles of the transfer function in the complex plane allows a qualitative interpretation of the behavior of the frequency response with varying frequency. This geometric interpretation is easily generalized to situations with any number of zeros and poles. It proves useful for the understanding of the general behavior of filters. The study begins here with the simplest system, the first order system. Then the second order circuit system is presented thoroughly. The logarithmic Bode representation of the frequency gain is introduced and its advantages demonstrated. The quality factor Q of a resonant circuit is defined.

2.1 First Order System. R, C Circuit

Consider the electrical circuit consisting of a resistor and a capacitor in series (Fig. 2.1). The circuit is powered by an internal resistance-free generator of electromotive force $e(t)$. The charge on one plate of the capacitor is written q, and the voltage across the capacitor noted $v(t) = \frac{q(t)}{C}$.

The generalized Ohm law writes:

$$R\frac{\mathrm{d}q}{\mathrm{d}t} + \frac{q}{C} = e(t). \tag{2.1}$$

With a system point of view, we write $e(t)$ as the input variable and $v(t)$ as the output variable.

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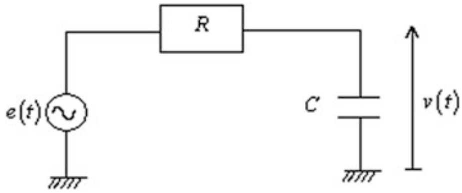


Fig. 2.1 First order system; R, C circuit

2.1.1 Transfer Function

The Eq. (2.1) can be written in the form of an operator acting on q:

$$\left(R\frac{\mathrm{d}}{\mathrm{d}t}+\frac{1}{C}\right)q=e(t). \tag{2.2}$$

This system is linear and time invariant. According to the fundamental result shown in Chap. 1, when $e(t)$ has the form e^{st}, the charge $q(t)$ on a plate of the capacitor and the voltage $v(t)$ across it will have the same exponential form. This can be checked:

Posing $e(t)=\mathrm{e}^{st}$ and looking for $q(t)$ in the form: $q(t)=B\mathrm{e}^{st}$.

Replacing its expression in Eq. (2.2) we have:

$$RB\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{st}+\frac{B}{C}\mathrm{e}^{st}=\mathrm{e}^{st}. \tag{2.3}$$

By simplifying by e^{st}, we see that the proposed solution is valid if the following relationship is satisfied:

$$\left(Rs+\frac{1}{C}\right)B=1. \tag{2.4}$$

or

$$B=\frac{1}{\left(Rs+\frac{1}{C}\right)}=\frac{C}{RCs+1}. \tag{2.5}$$

The voltage across the capacitor (system output variable) is given by:

$$v(t)=\frac{q}{C}=\frac{1}{RCs+1}\mathrm{e}^{st}=H(s)\mathrm{e}^{st}. \tag{2.6}$$

We notice that e^{st} is eigenfunction of the system and that $H(s)$ is its transfer function. The circuit transfer function $H(s)$ is thus written

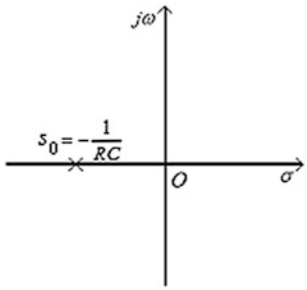


Fig. 2.2 Pole of the transfer function in the s plane

$$H(s) = \frac{1}{RCs + 1}. \tag{2.7}$$

$H(s)$ is a rational fraction with a simple pole (a simple zero of the denominator) in (See Fig. 2.2)

$$s_0 = -\frac{1}{RC}. \tag{2.8}$$

We can equivalently write $H(s)$ as

$$H(s) = \frac{-s_0}{s - s_0}. \tag{2.9}$$

The presence of a single simple pole is the reason for the first-order system name applying to this circuit.

2.1.2 Frequency Response

The frequency response is a particular case of the transfer function. In the function $H(s)$, the variable s is a complex number that will be written in the form: $s = \sigma + \mathrm{j}\omega$. s belongs to the complex plane. With reference to the Laplace transformation detailed in Chap. 9, the s plane is also called Laplace plane. This plane is identified by the real axis σ and the imaginary axis $\mathrm{j}\omega$.

$$v(t) = \frac{1}{RC(\sigma + \mathrm{j}\omega) + 1} \mathrm{e}^{\sigma t} \mathrm{e}^{\mathrm{j}\omega t}. \tag{2.10}$$

If σ = 0, that is, for a monochromatic input signal $e(t) = \mathrm{e}^{\mathrm{j}\omega t}$:

$$v(t) = \frac{1}{1 + \mathrm{j}RC\omega} \mathrm{e}^{\mathrm{j}\omega t} = H(\omega)\mathrm{e}^{\mathrm{j}\omega t} \tag{2.11}$$

The frequency response is:

$$H(\omega) = \frac{1}{1 + \mathrm{j}RC\omega} \tag{2.12}$$

The angular frequency ω is related to the frequency f by the relationship $\omega = 2\pi f$.

We see, of course, that $\mathrm{e}^{\mathrm{j}\omega t}$ is also eigenfunction of the system. The frequency response $H(\omega)$ (also called complex gain of the filter) is the transfer function $H(s)$ evaluated on the imaginary axis $\sigma = 0$.

By showing the modulus $|H(\omega)|$ and the argument φ of the frequency response, we write:

$$v(t) = H(\omega)\mathrm{e}^{\mathrm{j}\omega t} = |H(\omega)|\mathrm{e}^{\mathrm{j}\varphi}\mathrm{e}^{\mathrm{j}\omega t} \tag{2.13}$$

Therefore, while the modulus of the input signal $\mathrm{e}^{\mathrm{j}\omega t}$ is 1, the output signal modulus is $|H(\omega)|$. It appears that the modulus of the frequency response is the gain in amplitude of the signal passing through the filter. The phase shift φ of the output signal relative to the input signal is the argument of the complex gain $H(\omega)$. The magnitude and phase are functions of ω in the general case.

Note: For convenience, the function $H(\omega)$ is called frequency response, although this function is expressed as a function of the angular frequency ω and not of the frequency f.

For the variation of the gain as a function of the frequency f, we replace ω by $2\pi f$ in the expression of $H(\omega)$.

As noted above, the function $\mathrm{e}^{\mathrm{j}\omega t}$ is the system eigenfunction but the function $\cos \omega t = \frac{\mathrm{e}^{\mathrm{j}\omega t} + \mathrm{e}^{-\mathrm{j}\omega t}}{2}$, linear combination of two eigenfunctions, is not. The system response for a cosine input is searched as follows:

If the electromotive force $\mathrm{e}^{\mathrm{j}\omega x}$ has the form $e(t) = \cos \omega t$, due to the linearity of the system, we can write the answer in the form:

$$v(t) = \frac{1}{2}\left(H(\omega)\mathrm{e}^{\mathrm{j}\omega t} + H(-\omega)\mathrm{e}^{-\mathrm{j}\omega t}\right) \tag{2.14}$$

$$v(t) = \frac{1}{2}\left(\frac{1}{1 + \mathrm{j}RC\omega}\mathrm{e}^{\mathrm{j}\omega t} + \frac{1}{1 - \mathrm{j}RC\omega}\mathrm{e}^{-\mathrm{j}\omega t}\right) = \frac{1}{2}\left(\frac{1}{1 + \mathrm{j}RC\omega}\mathrm{e}^{\mathrm{j}\omega t}\right) + c.c. \tag{2.15}$$

c.c. is written to describe a complex conjugate of the previous term within the equation. The sum of a complex number and of its complex conjugate is equal to twice its real part, the following applies:

$$v(t) = \Re e\left(\frac{1}{1 + \mathrm{j}RC\omega}\mathrm{e}^{\mathrm{j}\omega t}\right) \tag{2.16}$$

The notation $\Re e()$ means that we must take the real part of the complex expression. It comes:

$$\begin{aligned} v(t) &= \Re e\left\{\left(\frac{1-\mathrm{j}RC\omega}{1+R^2C^2\omega^2}\right)(\cos\omega t+\mathrm{j}\sin\omega t)\right\} \\ &= \frac{1}{1+R^2C^2\omega^2}(\cos\omega t+RC\omega\sin\omega t). \end{aligned} \tag{2.17}$$

We can rewrite this result in the form:

$$v(t) = \frac{1}{\sqrt{1+R^2C^2\omega^2}}\left(\frac{1}{\sqrt{1+R^2C^2\omega^2}}\cos\omega t+\frac{RC\omega}{\sqrt{1+R^2C^2\omega^2}}\sin\omega t\right), \tag{2.18}$$

or in another form:

$$v(t) = \frac{1}{\sqrt{1+R^2C^2\omega^2}}\cos(\omega t+\varphi), \tag{2.19}$$

with

$$\cos\varphi = \frac{1}{\sqrt{1+R^2C^2\omega^2}} \quad \text{and} \quad \sin\varphi = \frac{-RC\omega}{\sqrt{1+R^2C^2\omega^2}}, \tag{2.20}$$

and then $\tan\varphi = -RC\omega$.

Behavior of the solution at low and high frequencies

At low frequencies, that is to say, when $RC\omega \ll 1$, we see on the solution (2.19) that $v(t) \cong \cos\omega t$. The output signal is in phase with the input signal and has equal amplitude.

At high frequency, when $RC\omega \gg 1$, the solution (2.19) becomes: $v(t) \cong \frac{1}{RC\omega}\sin\omega t$.

The output signal is in quadrature with the input signal with a phase shift $\varphi \cong -\frac{\pi}{2}$ and its amplitude decreases with frequency as $\frac{1}{\omega}$.

Note: Conciseness of the results when expressed in the form of complex exponentials will be compared to the heaviness from those expressed in sine and cosine.

2.1.3 Graphic Representation of the Frequency Response

Since $H(\omega) = \frac{1}{1+\mathrm{j}RC\omega}$, the modulus is:

$$|H(\omega)| = \frac{1}{\sqrt{1+R^2C^2\omega^2}}, \tag{2.21}$$

and phase

$$\varphi = -\mathrm{Arg}(RC\omega) \tag{2.22}$$

In Fig. 2.3, are represented the modulus and phase of $H(\omega)$.

For the value ω_c of ω such that $RC\omega_c = 1$, the value of the gain modulus is $\frac{1}{\sqrt{2}}$.

Using its value in decibels: $H_{\mathrm{dB}} = 20\log_{10}(|H(\omega_c)|) = 20\log_{10}\left(\frac{1}{\sqrt{2}}\right) = -3\,\mathrm{dB}$.

Frequency $f_c = \frac{\omega_c}{2\pi} = \frac{1}{2\pi RC}$ is called the -3 dB cutoff frequency.

It is seen in Fig. 2.3b that the phase variation range goes from $\frac{\pi}{2}$ to $-\frac{\pi}{2}$.

Bode representation

Scale in decibels

In the Bode representation the magnitudes logarithm are represented. As mentioned above, the decibel value of a quantity A is $A_{\mathrm{dB}} = 20\log_{10} A$. This unit of measure was introduced by G. Bell to describe the acoustic sensitivity of the human ear (hence the name of this unit). The sensitivity of the ear is logarithmic: if the intensity of a sound is multiplied by 10, the ear feels a multiplication by 2. If the intensity is multiplied by 100, the ear feels a multiplication by 4. This physiological property allows the ear to hear correctly loud sounds, but remain sensitive to very low sounds. Moreover, as will be discussed in Chap. 3, the note of a musical instrument is accompanied by the presence of harmonics whose frequencies are multiples of the fundamental frequency. The amplitudes of these harmonics are specific to each instrument. They can be several tens of times lower than that of the fundamental component. As the ear analyzes the sounds from frequency, its logarithmic sensitivity somehow enhances the amplitude of low harmonics. This allows it to be physiologically sensitive to harmonics, so to the musicality of the instrument. It is important to remember that the representation in logarithm reinforces the low values of a variable relatively to strong values. This property is exploited in the Bode representation with which we may monitor small changes of

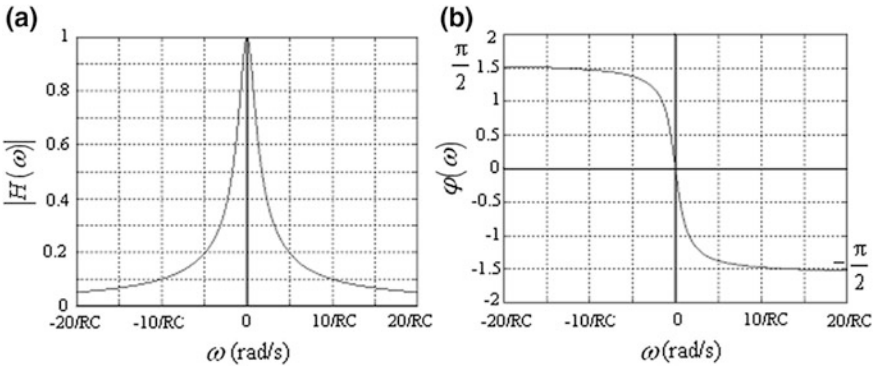


Fig. 2.3 Frequency response of R C circuit. **a** Modulus. **b** Phase

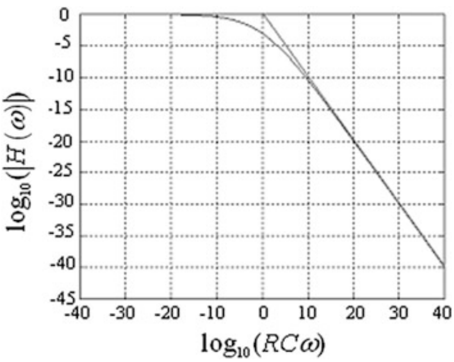


Fig. 2.4 Log–log plot of gain magnitude (first order system)

the variable values, whereas in linear representation, they would have been undetectable. This representation has better dynamics. This explains why the gain of the filters is most often plotted in dB.

Another quality of the logarithmic representation is that a variation with frequency in power law appears as a straight line whose slope gives the value of the power law coefficient.

By definition, the decibel value of the frequency response is equal to 20 times the base 10 logarithm of the frequency response modulus.

$$H_{\mathrm{dB}} = 20\log_{10}(|H(\omega)|). \tag{2.23}$$

Assuming that at high frequencies, the system has an asymptotic behavior of the form $|H(\omega)| \simeq \omega^n$, then $H_{\mathrm{dB}} = 20\log_{10}\omega^n = n\,20\log_{10}\omega$. In a logarithmic representation $H_{\mathrm{dB}} = f(20\log_{10}\omega)$, the variation is linear.

Figure 2.4 shows the gain in dB of the first order filter. Note the linear asymptotic behavior of the high-frequency curve. The asymptote passes through the point (0, 0), that is to say, for the x-axis value $\omega = \frac{1}{RC}$. The slope of the line is -1, reflecting the asymptotic gain as $\frac{1}{\omega}$ (Fig. 2.4). $|H(\omega)|$ decreases by 20 decibels per decade (a decade corresponds to a multiplication of the frequency by a factor of 10). This decrease is also -6 dB per octave (the octave is defined in music as the interval between two notes when the frequency of a note is twice that of the other. For example, the frequency of the note C is multiplied by 2 when going on a piano keyboard from a C to a C immediately above).

2.1.4 Geometric Interpretation of the Variation of the Frequency Response

It has been shown above that the transfer function is: $H(s) = \frac{-s_0}{s-s_0}$, with $s_0 = -\frac{1}{RC}$.

The frequency response is:

$$H(\omega) = \frac{-s_0}{\mathrm{j}\omega - s_0}. \tag{2.24}$$

In the complex plane $s = \sigma + j\omega$. The point of the plane corresponding to the real pole $s_0 = -\frac{1}{RC}$ is noted on Fig. 2.5. The point M is the point $\mathrm{j}\omega$ representative of the monochromatic signal to the frequency ω. The complex number in the denominator of $H(\omega)$ can be associated to the vector $\overrightarrow{PM}$. The modulus of $H(\omega)$ is inversely proportional to the length PM of that vector:

$$|H(\omega)| = \frac{|-s_0|}{PM} = \frac{1}{RC}\frac{1}{PM}. \tag{2.25}$$

Using the Pythagorean theorem we write $PM = \sqrt{\omega^2 + \frac{1}{R^2C^2}}$.

We find the variation in function of the frequency of the modulus of $H(\omega)$ according to the variation in length of the segment PM when the point M scans the vertical axis $\sigma = 0$ from $-\mathrm{j}\infty$ (frequency $-\infty$) to $+\mathrm{j}\infty$ (frequency $+\infty$).

For very high negative frequencies the segment PM is very large, and its inverse is very small. Thus $|H(\omega)|$ is very small. When the frequency decreases in absolute value to the zero frequency, the segment PM decreases, and $|H(\omega)|$ increases.

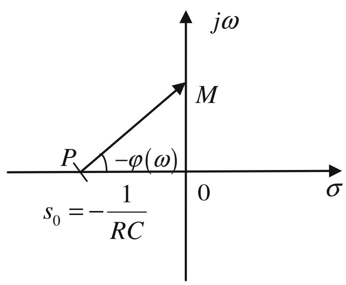
The segment PM is minimal for $\omega = 0$ and its inverse $|H(\omega)|$ is maximal. The gain will decrease continuously when ω increases from zero, the segment PM continuously growing. As shown on Fig. 2.6. Since the phase of the output signals is equal to the argument of $H(\omega)$,

$$\varphi(\omega) = \mathrm{Arg}(H(\omega)) = \mathrm{Arg}(-s_0) - \mathrm{Arg}(\mathrm{j}\omega - s_0).$$

s_0 being real and negative, we have $\varphi(\omega) = -\mathrm{Arg}(\mathrm{j}\omega - s_0)$.

The argument of $\mathrm{j}\omega - s_0$ is equal to the angle formed by the vector $\overrightarrow{PM}$ with the horizontal axis. When the frequency is largely negative this angle is close to $-\frac{\pi}{2}$, the phase of $H(\omega)$ (the opposite to that angle) is then close to $\frac{\pi}{2}$. The change of phase with frequency is shown Fig. 2.6.

Fig. 2.5 Vector $\overrightarrow{PM}$ situation for a given frequency



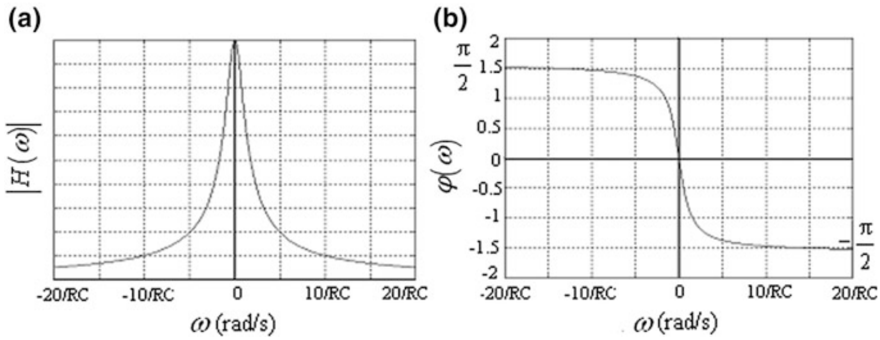
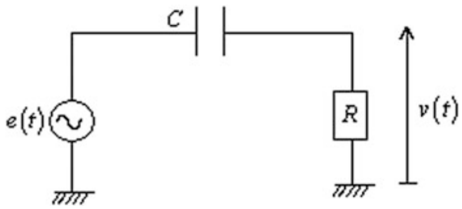


Fig. 2.6 Frequency response of R C circuit after geometric interpretation. **a** Modulus. **b** Phase

Fig. 2.7 R C Circuit with output taken at resistor terminals



2.1.5 R, C Circuit with Output on the Resistor Terminals

This system is a second example of a first order system. The circuit is identical to that of Sect. 1.1 but the output voltage is taken at the terminals of the resistor (Fig. 2.7). We have the following diagram:

The calculation of the charge across the capacitor is the same as in Sect. 2.1.1. When $e(t) = \mathrm{e}^{st}$ we have again:

$$q(t) = \frac{C}{RCs+1}\mathrm{e}^{st}. \tag{2.26}$$

$$v(t) = R\frac{\mathrm{d}q}{\mathrm{d}t} = \frac{RCs}{RCs+1}\mathrm{e}^{st} = H(s)\mathrm{e}^{st}. \tag{2.27}$$

The transfer function is in this case:

$$H(s) = \frac{RCs}{RCs+1} = -s_0\frac{RCs}{s-s_0} = \frac{s}{s-s_0}. \tag{2.28}$$

The transfer function has a zero in $s = 0$ and a pole in $s_0 = -\frac{1}{RC}$.

Geometric interpretation of the variation of gain with frequency:
We have:

$$H(\omega) = \frac{j\omega}{j\omega - s_0}. \tag{2.29}$$

As can be seen in Fig. 2.5, the gain modulus is equal to the ratio of two segments:

$$|H(\omega)| = \frac{OM}{PM}. \tag{2.30}$$

As ω varies, the point M scans upward the axis $\sigma = 0$.

When $|\omega|$ is very large, the lengths of the segments OM and PM are very slightly different, the gain is close to 1. When ω is close to zero, the numerator becomes small while the denominator remains finite. The gain in amplitude $|H(\omega)|$ is close to zero.

The phase is the argument of the numerator of $H(\omega)$ minus the argument of its denominator:

$$\varphi(\omega) = \mathrm{Arg}(j\omega) - \mathrm{Arg}(j\omega - s_0). \tag{2.31}$$

$\mathrm{Arg}(j\omega)$ equals $-\frac{\pi}{2}$ when $\omega < 0$ and equals $\frac{\pi}{2}$ if $\omega > 0$ (there is a π jump when ω passes through zero). As seen above, $-\mathrm{Arg}(j\omega - s_0)$ varies from $\frac{\pi}{2}$ to $-\frac{\pi}{2}$ when ω varies from $-\infty$ to $+\infty$. The variations of the gain and phase with ω are shown in Fig. 2.8.

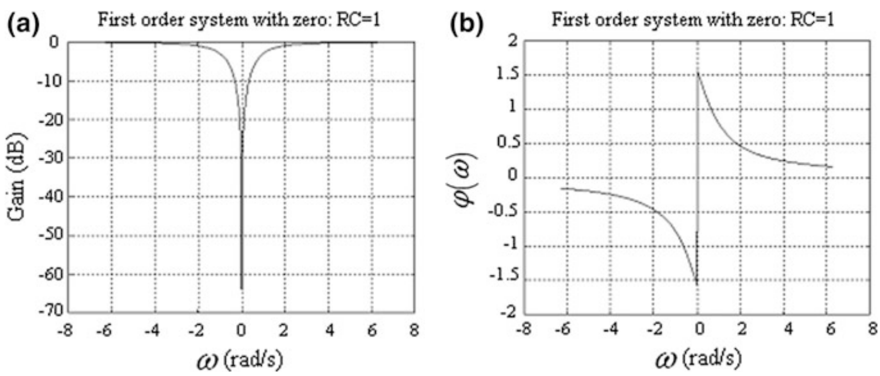


Fig. 2.8 Frequency gain for second R C circuit. **a** Modulus. **b** Phase

2.2 Second Order System. R, L, C Series Circuit

The emf $e(t)$ is applied to the terminals of a circuit composed of an inductor L, a resistor R and a capacitor C in series (Fig. 2.9). As above, the electric charge on a plate of the capacitor is denoted by q, and $v(t)$ is the voltage across the capacitor.

Generalized Ohm's law takes the form:

$$L\frac{\mathrm{d}^2q}{\mathrm{d}t^2} + R\frac{\mathrm{d}q}{\mathrm{d}t} + \frac{q}{C} = e(t). \tag{2.32}$$

2.2.1 *Transfer Function*

This system is linear, invariant by translation in time. The circuit transfer function $H(s)$ is obtained by taking $e(t) = \mathrm{e}^{st}$ for excitation and seeking $q(t)$ of the form $q(t) = B\mathrm{e}^{st}$:

$$LB\frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathrm{e}^{st} + RB\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{st} + \frac{B}{C}\mathrm{e}^{st} = \mathrm{e}^{st}. \tag{2.33}$$

That is to solve the equation

$$\left(Ls^2 + Rs + \frac{1}{C}\right)B = 1. \tag{2.34}$$

It is necessary to have the equality $B = \frac{1}{Ls^2 + Rs + \frac{1}{C}} = \frac{C}{LCs^2 + RCs + 1}$.

The voltage across the capacitor is given by:

$$v(t) = \frac{q(t)}{C} = \frac{1}{LCs^2 + RCs + 1}\mathrm{e}^{st} = H(s)\mathrm{e}^{st}. \tag{2.35}$$

The system transfer function is therefore:

$$H(s) = \frac{1}{LCs^2 + RCs + 1}. \tag{2.36}$$

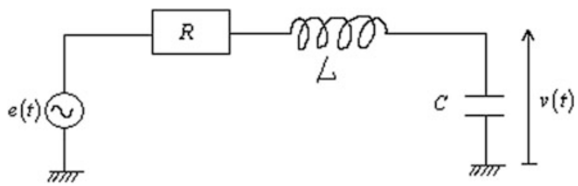


Fig. 2.9 Second order circuit; R L C in series

The transfer function is again a rational fraction which must initially determine the poles. The denominator is a polynomial in s. A general property of polynomials with complex coefficients is that they always have roots. These roots belong to the field of complex numbers. In addition, another property of polynomials is that when all the coefficients of the various powers of the variable s are real, the roots are either real or come in complex conjugate pairs.

Search of the Poles of the Transfer Function
The polynomial being of second degree, he always has two roots which will be distinct or multiple. For this reason, this circuit is called a second order filter.

The transfer function has the general form:

$$H(s) = \frac{1}{LCs^2 + RCs + 1} = \frac{1}{LC}\frac{1}{(s - s_1)(s - s_2)}. \tag{2.37}$$

Analysis of the roots of the quadratic polynomial $LCs^2 + RCs + 1$:
The discriminant of the polynomial is:

$$\Delta = R^2C^2 - 4LC. \tag{2.38}$$

The roots of the polynomial are noted s_1 and s_2.

- If $\Delta > 0$, $s_{1,2} = \dfrac{-RC \pm \sqrt{R^2C^2 - 4LC}}{2LC} = -\dfrac{R}{2L} \pm \sqrt{\dfrac{R^2}{4L^2} - \dfrac{1}{LC}}$. (2.39)

The two roots are real.

- If $\Delta = 0$, $s_1 = s_2 = -\dfrac{R}{2L}$, the polynomial has a double real root. (2.40)

- If $\Delta < 0$, $s_{1,2} = -\dfrac{R}{2L} \pm \mathrm{j}\sqrt{\dfrac{1}{LC} - \dfrac{R^2}{4L^2}}$, the two roots are complex conjugate: (2.41)

Writing

$$\omega_0 = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}, \tag{2.42}$$

we have:

$$s_{1,2} = -\frac{R}{2L} \pm j\omega_0 \tag{2.43}$$

2.2.2 Second Order System Frequency Response

The poles of the transfer function will condition the frequency response of the system $H(\omega)$, response to a monochromatic input signal of the form $e^{j\omega t}$.

Simply replacing s by $j\omega$ in the expression of $H(s)$, we have

$$H(\omega) = \frac{1}{-LC\omega^2 + jRC\omega + 1} = \frac{1}{LC}\frac{1}{(j\omega - s_1)(j\omega - s_2)}. \tag{2.44}$$

Note that $H(0) = 1$. Calculation programs like Matlab easily enable graphical representation of the modulus and phase of $H(\omega)$.

2.2.3 Geometric Interpretation of the Variation of the Frequency Response

It is interesting to further develop a geometric argument to interpret the variation of the frequency response. Its modulus is:

$$|H(\omega)| = \frac{1}{LC}\frac{1}{|j\omega - s_1||j\omega - s_2|}. \tag{2.45}$$

Having placed the poles s_1 (point P_1) and s_2 (point P_2) in the complex plane $(\sigma, j\omega)$, we see that the modulus of $H(\omega)$ is inversely proportional to the lengths of segments joining point M (representing jω) to the points P_1 and P_2.

$$|H(\omega)| = \frac{1}{LC}\frac{1}{MP_1 MP_2}. \tag{2.46}$$

The phase is given by the sum of the angles made by the vectors $\overrightarrow{P_1M}$ and $\overrightarrow{P_2M}$ with the x-axis:

$$\varphi(\omega) = \mathrm{Arg}(H(\omega)) = -\mathrm{Arg}(j\omega - s_1) - \mathrm{Arg}(j\omega - s_2). \tag{2.47}$$

One can thus deduce qualitatively the following variations of gain and phase:

- If $\Delta > 0$, poles s_1 and s_2 lie on the real axis $\omega = 0$ (Fig. 2.10).

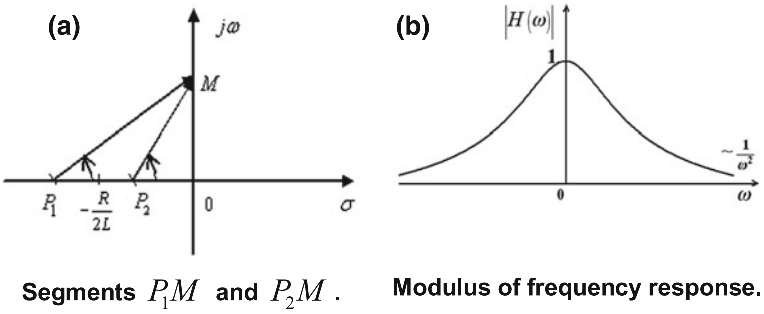


Fig. 2.10 Geometric interpretation in case of two real poles. **a** Poles situation. **b** Gain modulus

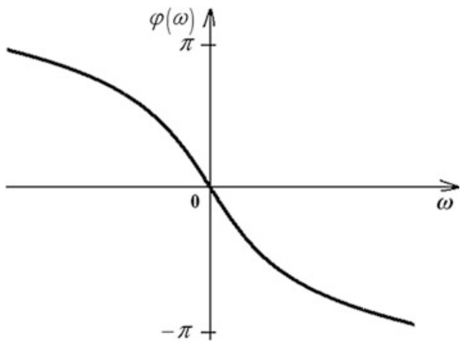


Fig. 2.11 Phase variation given by geometric interpretation

We see in Fig. 2.10 that the maximum gain value is obtained for zero frequency, value of ω for which the segments P_1M and P_2M are minimal. The gain decreases as $\frac{1}{\omega^2}$ when $|\omega| \to \infty$, each of the two segments P_1M and P_2M growing like $|\omega|$. The circuit behaves as a low pass filter.

When ω is largely negative, angles of the two vectors $\overrightarrow{P_1M}$ and $\overrightarrow{P_2M}$ with the horizontal are each approximately $-\frac{\pi}{2}$; phase will be π.

When ω increases, M scans vertically the axis $\sigma = 0$ and angles vary from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$ (Fig. 2.11). They will be 0 for $\omega = 0$, the phase will be zero. Then as the angles increase toward $\frac{\pi}{2}$ the phase tends toward $-\pi$.

- If $\Delta = 0$, both poles are merged on the real axis (Fig. 2.12). The discussion is similar to the previous case and the system still has a low-pass filter behavior (Fig. 2.13).
- If $\Delta < 0$, the two poles are complex conjugates. We note H_1 and H_2 the projections of P_1 and P_2 on the axis jω (Fig. 2.14). For large negative values of ω, we have the same behavior as before, the segments P_1M and P_2M are large and the modulus $|H(\omega)|$ very small, and the phase tends toward π (Figs. 2.14, 2.15 and 2.16).

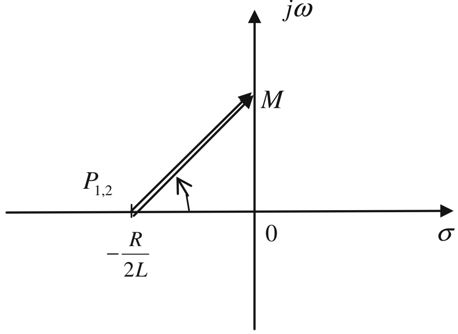


Fig. 2.12 Two poles merged on real axis

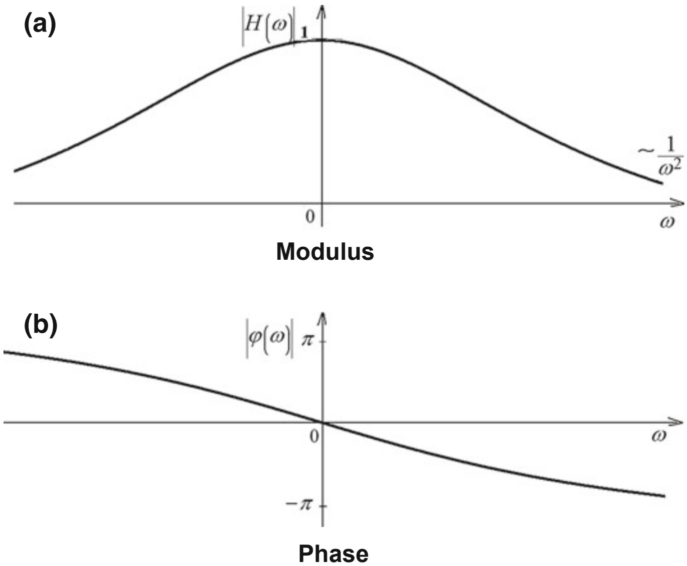
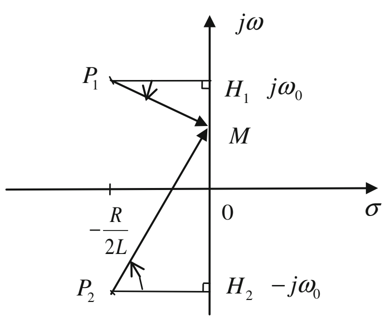


Fig. 2.13 Frequency gain in case of a double real pole. **a** Magnitude. **b** Phase

Fig. 2.14 Vectors P_1M and P_2M situation for a given frequency



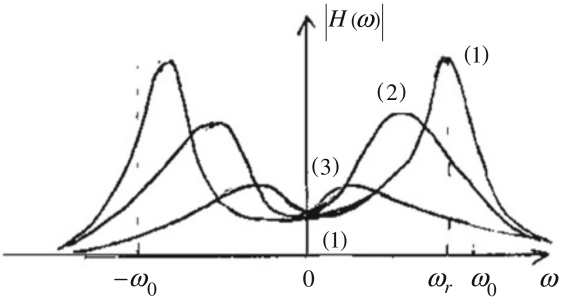


Fig. 2.15 Gain magnitudes for different damping situations

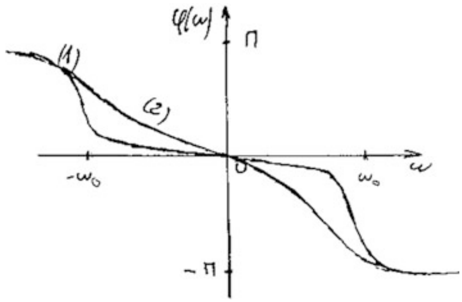


Fig. 2.16 Gain phases for different damping situations

- When ω increases, a first maximum of $|H(\omega)|$ will occur when the product of the lengths of the segments P_1M and P_2M will reach a minimum. This will occur to a first intermediate position of M between H_2 and O.

The more points P_1 and P_2 will be close to the imaginary axis, that is to say, the more $\frac{R}{2L}$ will be small compared to ω_0, the more segments P_1M and P_2M can become smaller and $|H(\omega)|$ can become great. The resonance pulsation ω_r for which $|H(\omega)|$ is maximal will be closer to ω_0 when the points P_1 and P_2 are close to the imaginary axis.

$$|H(\omega)|_{\max} \cong |H(\omega_0)| = \frac{1}{LC}\frac{1}{H_1P_1}\frac{1}{H_1P_2}. \tag{2.48}$$

The quantity $\frac{R}{2L}$ characterizes the damping of the circuit. Curves 1, 2 and 3 in Fig. 2.15 show the trend of the gain when the damping is increasing (with respect to ω_0). Thus, it is to remember that as the pole is closer to the vertical axis, the resonance is sharper and the resonance frequency nearer to ω_0.

Regarding the phase, it is found that when the pole is close to the vertical axis, the angle of the vector $\overrightarrow{P_1M}$ with horizontal changes abruptly from a value close to $-\frac{\pi}{2}$ to a value close to $\frac{\pi}{2}$ when ω passes through resonance (Fig. 2.16). In the case of strong resonance, phase starts from π and varies from π to 0 when ω passes the value $-\omega_r$ where the phase passes to $-\pi$. In the case of a damped system (pole farther from the vertical axis) angles vary more gradually.

It may be noted to put an end to this discussion that, as before, the gain decreases as $\frac{1}{\omega^2}$ when $|\omega| \to \infty$, both segments P_1M and P_2M in the denominator of the frequency response increasing as $|\omega|$.

On the following graphs showing the magnitude (Fig. 2.17) and phase (Fig. 2.18) of the gain, in the case where $L = 0.1$, $C = 0.1$ and where R was varied by taking the values 0.1, 0.3, 0.5, 0.7, 0.9.

The module is shown in linear scale:

$\frac{1}{RC}$ represents roughly the half width of the modulus of $H(\omega)$.

Phase varies from π to $-\pi$.

The resonance frequency ω_r which is the abscissa of the maximum of the frequency response modulus, is analytically determined by annulling the derivative

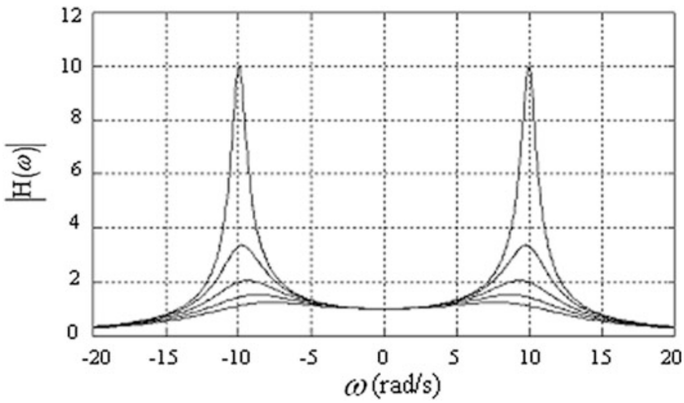


Fig. 2.17 Numerical simulations: gain magnitudes for different damping

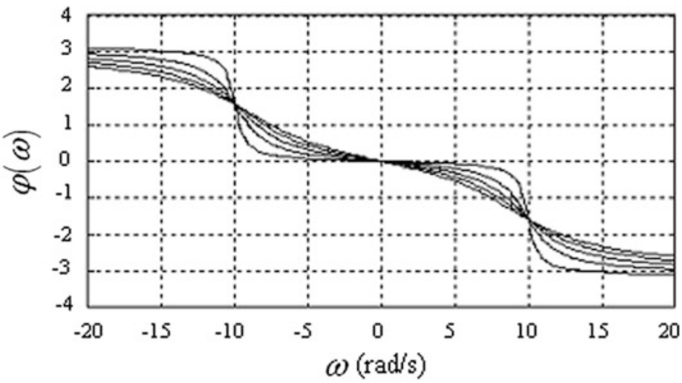


Fig. 2.18 Numerical simulations: gain phases for different damping

of $|H(\omega)|$. The denominator of the frequency responsemodule includes the sum of squares of the real and the imaginary parts:

$$|H(\omega)| = \sqrt{\frac{1}{(1 - LC\omega^2)^2 + R^2C^2\omega^2}}. \tag{2.49}$$

The resonant frequency ω_r is the frequency for which $\frac{\mathrm{d}|H(\omega)|}{\mathrm{d}\omega} = 0$.

This amounts to calculating the solutions of the equation canceling the derivative of the denominator. It comes:

$$\omega_r = \pm\sqrt{\frac{1}{LC} - \frac{R^2}{2L^2}}. \tag{2.50}$$

There are two solutions with opposite signs. Only the positive frequency is significant for real signals. It can be seen on the positive root that as the resistance R increases it causes the decrease of the resonant frequency, as it was anticipated qualitatively.

To calculate the filter gain at the resonance, this root is reported in the gain expression. It comes:

$$|H(\omega_r)| = \sqrt{\frac{1}{\left(1 - LC\omega_r^2\right)^2 + R^2C^2\omega_r^2}} = \frac{1}{RC\omega_0}. \tag{2.51}$$

Where ω_0 is given by (2.42). This result is remarkable for its simplicity.

2.2.4 Bode Representation of the Gain

Figure 2.19 shows the variation of $20\log_{10}(|H(\omega)|) = f\left(20\log_{10}\frac{\omega}{\omega_r}\right)$ for the following values of the system parameters: $R = 1\ \Omega$, $L = 10^{-4}$ H, $C = 10^{-6}$ F.

This gives the resonant frequency $\omega_r = 9.975 \times 10^4$ rad/s. The gain for the resonance frequency is approximately equal to 20 dB. The theoretical gain in decibels for the resonance frequency is calculated from the formula (2.51). It is

$$20\log_{10}|H(\omega_r)| = 20\log_{10}\frac{1}{RC\omega_0} = 20.01\ \mathrm{dB}. \tag{2.52}$$

The system resonant frequency $\omega_r = 9.975 \times 10^4$ rad/s is slightly lower than the frequency $\omega_0 = 9.9875 \times 10^4$ rad/s, imaginary part of the positive pole frequency. Resonance is sharp.

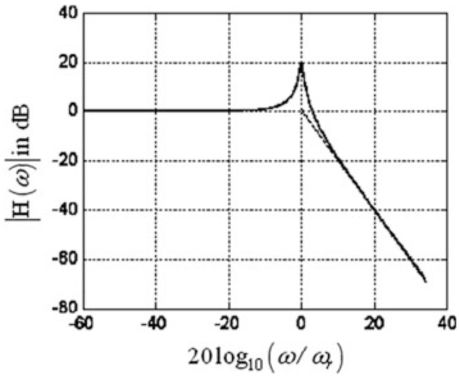


Fig. 2.19 Log-log plot of gain magnitude (second order system)

Note the linear behavior of the curve at high frequencies. As seen in Fig. 2.19, at high frequencies $|H(\omega)|$ decreases of 40 decibels per decade (when the frequency is multiplied by 10), as characteristic of the decay in $\frac{1}{\omega^2}$. This decrease corresponds to -12 dB per octave (when the frequency is multiplied by 2).

The asymptotic line to the high frequency curve (dotted line in Fig. 2.19) passes through the point (0, 0), that is to say, for the abscissa value $\omega = \omega_r$. Please note that this is only true in the case of sharp resonance that is specified in the following paragraph.

2.3 Case of Sharp Resonance

We have seen that in the case of a sharp resonance, the resonance frequency which corresponds to the maximum of $|H(\omega)|$ is near ω_0. We can use in this case an approximate expression of $|H(\omega)|$ in the vicinity of the resonance. Geometrically, when ω is near ω_0, we allocate all of the variation of the modulus $|H(\omega)|$ to the variation of the segment MP_1. In the scheme of this approximation, the gain is maximum when M is in H_1:

$$|H(\omega)|_{\max} \cong |H(\omega_0)| = \frac{1}{LC}\frac{1}{H_1P_1}\frac{1}{H_1P_2}. \tag{2.53}$$

We have approximately $H_1P_2 \cong H_1H_2$, then:

$$|H(\omega)|_{\max} \cong |H(\omega_0)| = \frac{1}{LC}\frac{1}{H_1P_1}\frac{1}{H_1H_2}. \tag{2.54}$$

Under this approximation of sharp resonance, as $H_1P_1 = \frac{R}{2L}$ and $H_1H_2 = 2\omega_0$, we have then:

$$|H(\omega)|_{\max} \cong \frac{1}{LC}\frac{2L}{R}\frac{1}{2\omega_0} = \omega_0^2\frac{2L}{R}\frac{1}{2\omega_0} = \frac{L\omega_0}{R}. \tag{2.55}$$

Bandwidth at −3 dB of the resonator

Noting M_1 the point on the imaginary axis as $H_1M_1 = H_1P_1$ and ω_1 the corresponding angular frequency, we have

$$\begin{aligned}|H(\omega_1)| &= \frac{1}{LC}\frac{1}{M_1P_1}\frac{1}{M_1P_2} \cong \frac{1}{LC}\frac{1}{\sqrt{(M_1H_1)^2+(H_1P_1)^2}}\frac{1}{H_1P_2}\\ &= \frac{1}{LC}\frac{1}{\sqrt{2(H_1P_1)^2}}\frac{1}{H_1H_2}.\end{aligned} \tag{2.56}$$

Therefore

$$|H(\omega_1)| \cong \frac{1}{\sqrt{2}}|H(\omega)|_{\max}. \tag{2.57}$$

Expressing this ratio in decibels:

$$\begin{aligned}|H(\omega_1)|_{\text{dB}} &= 20\log_{10}|H(\omega_1)| \cong 20\log_{10}|H(\omega)|_{\max} + 20\log_{10}\frac{1}{\sqrt{2}},\\ |H(\omega_1)|_{\text{dB}} &= |H(\omega)|_{\max(\text{dB})} - 3\,\text{dB}.\end{aligned} \tag{2.58}$$

At point M_2 (pulsation ω_2) symmetrical of M_1 with respect to H_1, the attenuation is also 3 dB relatively to the maximum gain of the filter. Bandwidth at −3 dB is then as follows:

$$\Delta\omega = \omega_2 - \omega_1 = 2H_1P_1 = \frac{R}{L}.$$

2.4 Quality Factor Q

We name Quality factor Q the ratio

$$Q = \frac{\omega_0}{\Delta\omega} \tag{2.59}$$

The sharper the resonance, the smaller $\Delta\omega$ and the higher Q.

In this case:

$$\omega_0 = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} \quad \text{and} \quad Q = \frac{\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}}{\frac{R}{L}}. \tag{2.60}$$

Since the damping is small, the second term in the root can be neglected when compared to the first term. Then

$$Q \cong \sqrt{\frac{1}{LC}}\frac{L}{R} = \sqrt{\frac{L}{C}}\frac{1}{R}. \tag{2.61}$$

Or, writing approximately

$$LC\omega_0^2 = 1, \quad Q = \frac{L\omega_0}{R}. \tag{2.62}$$

It is noted that, in the case of sharp resonance, the value of Q is equal to the maximum gain at resonance. Indeed, it has been seen that $|H(\omega_r)| = \frac{1}{RC\omega_0}$. As in the case of sharp resonance, the relationship $LC\omega_0^2 = 1$ is approximately satisfied, it finally comes $|H(\omega_r)| = \frac{L\omega_0}{R} = Q$, as it had been shown geometrically in the preceding paragraph (Eq. (2.55)).

Decrease over time in the amplitude of the eigenfunctions corresponding to the values of the poles

The eigenfunctions of the resonant system for values of s equal to those of the poles have the form

$$\mathrm{e}^{s_{1,2}t} = \mathrm{e}^{-\frac{R}{2L}t \pm \mathrm{j}\omega_0 t} = \mathrm{e}^{-\frac{R}{2L}t}\mathrm{e}^{\pm \mathrm{j}\omega_0 t} \tag{2.63}$$

The amplitude of these functions varies with time as $\mathrm{e}^{-\frac{R}{2L}t}$. In a pseudoperiod $T_0 = \frac{2\pi}{\omega_0}$, this amplitude will vary by a factor

$$\mathrm{e}^{-\frac{R}{2L}T_0} = \mathrm{e}^{-\frac{R}{2L\omega_0}2\pi} = \mathrm{e}^{-\frac{\pi}{Q}}. \tag{2.64}$$

When the Q-factor is great compared to 1, we can perform a limited expansion of the exponential and write: $e^{-\frac{\pi}{Q}} \cong 1 - \frac{\pi}{Q} + \ldots$

In a pseudoperiod, the amplitudes of functions $\mathrm{e}^{s_1 t}$ and $\mathrm{e}^{s_2 t}$ decrease by a factor $\frac{\pi}{Q}$. It will be shown in the following that the following linear combination of these functions $\mathrm{e}^{s_{1,2}t}$ is the response of second order system in a very short pulse (Dirac pulse). This impulse response has the form:

$$h(t) = \frac{1}{LC}\frac{1}{(s_1 - s_2)}(\mathrm{e}^{s_1 t} - \mathrm{e}^{s_2 t})U(t). \tag{2.65}$$

In practice, in the case of complex conjugate poles, one measures the Q-factor from the decay of $h(t)$ during the pseudoperiod T_0.

Summary

The important first and second order electrical *R, L, C* circuit systems were studied in this chapter. The position of the poles of the transfer functions was used for qualitatively explaining the variation in frequency of the module and phase responses. This interpretation is fundamental in understanding the behavior of these filters and provides a generalized view of the frequency response of electronic systems. The Bode representation has been presented. The concept of quality factor used to characterize the properties of many physical systems was introduced.

Exercises

I. Consider the circuit composed of the series arrangement of a resistor $R = 100\ \Omega$, an inductor coil value $L = 0.01$ H, and a capacitance $C = 10^{-10}$ F. Note $e(t)$ the voltage across the assembly and $v(t)$ the voltage across the capacitor.

1. It is assumed that the emf $e(t)$ has the form $e(t) = \mathrm{e}^{st}$ where s is a complex number capacitor $(s = \sigma + \mathrm{j}\omega)$.

   (a) Give the expression of the voltage $v(t)$.
   (b) Give the expression of the filter transfer function. What are the poles of this transfer function? Represent the position of the poles.
   (c) Give the expression of the filter's frequency response. By a geometric argument based on the position of the poles, give the aspect of the variation of gain with frequency module.

2. Note that the transfer function can be written as a product of two terms of the first order $H(s) = H_1(s)H_2(s)$. From the variation of $|H_1(\omega)|$ with ω, give the $-3\,\mathrm{dB}$ bandwidth of the first filter. By noticing that $|H_2(\omega)|$ remains approximately constant in the vicinity of the resonance, give the bandwidth at -3 dB of $|H(\omega)|$.

II. Consider again the circuit including the elements R, L and C placed in series with the output at the resistor terminals this time. Show that the transfer function is in this case:

$$H(s) = \frac{RCs}{LCs^2 + RCs + 1}.$$

Locate the zeros of $H(s)$ in the complex plane. Show that the circuit does not allow the continuous to pass (the frequency response is zero at zero frequency). Can this system keep a resonator character? Show that the resonance frequency is equal to $\omega_{00} = \sqrt{\frac{1}{LC}}$ whatever damping. Explain qualitatively that the presence of the zero of the transfer function pushes the positive resonance

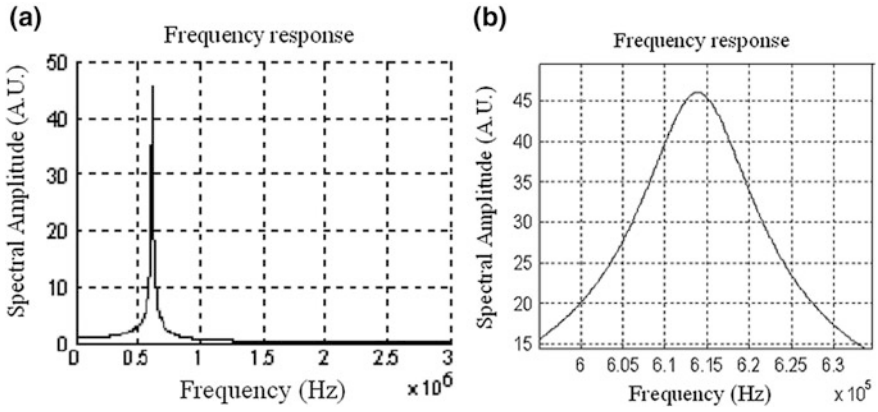


Fig. 2.20 Gain magnitude (**a**); with zoom in (**b**)

frequency toward higher frequencies, frequency as the negative pole frequency tends to decrease.
Which is the gain of the resonance filter? Show that the gain decreases at high frequencies.
Qualitatively, observe that the presence of the zero of the transfer function at $\omega = 0$ pushes the positive resonance frequency toward higher frequencies, while the negative pole tends to decrease that frequency. What is the filter gain at resonance? Show that the gain decreases as $\frac{1}{\omega}$ at high frequencies.

III. Create a circuit of the second order by arranging an inductor L, a resistor R and a capacitor C in series. The input signal is feeding the ensemble and the output signal is taken across the capacitance.

(A) The modulus of this filter frequency response is given by Fig. 2.20:

1. What is the value of the quality Q-factor of the circuit?
2. (a) Making the approximation of a sharp resonance, taking $R = 4.7\ \Omega$, evaluate L and C knowing that the resonant frequency is precisely $6.1389\ 10^5$ Hz.
   (b) Place the poles of the filter transfer function in the Laplace plane.

(B) The impulse response of that filter is given in Fig. 2.21.
Evaluate from these curves L and C the quality factor of the circuit, still taking $R = 4.7\ \Omega$.

Solution:

(A) In the graph of the frequency response we can estimate its maximum amplitude at about 46. In the course, it has been shown that the maximum amplitude is equal to the Q-factor. So we evaluate $Q = 46$.

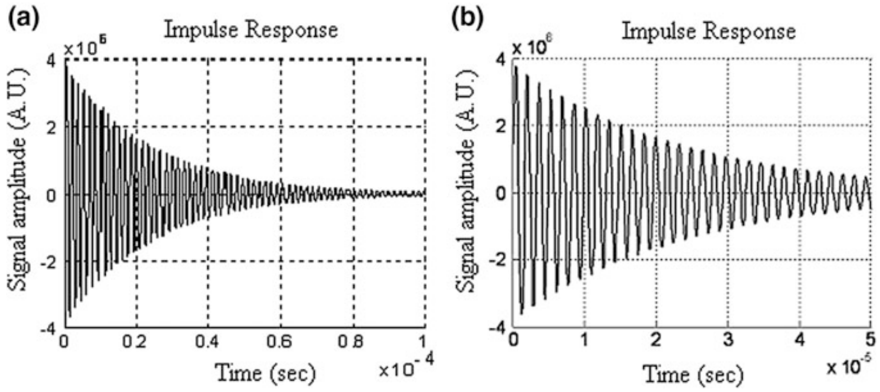


Fig. 2.21 Impulse response (**a**); with zoom in (**b**)

Second method for determining Q: $Q = \frac{\omega_0}{\Delta\omega} = \frac{f_0}{\Delta f}$ (Δf is the bandwidth at -3 dB). The amplitude at -3 dB is estimated to $\frac{46}{\sqrt{2}} = 32.5$.

On the graph of the frequency response the resonance frequency is seen to be $f_0 = 6.14 \times 10^5$ Hz. The frequencies for which the frequency response is attenuated by -3 dB are 6.21×10^5 Hz and 6.075×10^5 Hz, and It can be inferred that $Q = \frac{6.14}{(6.21-6.075)} = 45.5$, which value must be equal to the value given by the first method, the difference being due to uncertainties determinations on the graph. Since $Q = 45.5 = \frac{L\omega_0}{R}$, the resonance frequency in this case of sharp resonance is given by: $LC\omega_0^2 = 1$. It comes $L = 5.510^5$ H and $C = 1.2 \times 10^{-9}$ F.

(B) Determination from the impulse response: we measure graphically the pseudoperiod T_0 of the signal and we deduce $\omega_0 = \frac{2\pi}{T_0}$. In a pseudoperiod, the amplitude varies by the factor $e^{-\frac{\pi}{Q}}$. We deduce Q from it. We then calculate the constants L and C of the circuit knowing R = 4.7 Ω.

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